Phase Transitions in Holographic Superconductors of Type II
— Talk Notes —

Heinrich-Gregor Zirnstein
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These are my notes for a talk in the seminar "Holographic Methods in Condensed Matter Theory". The introduction to type II superconductors closely follows Ref. [3]. The discussion of the holographic approach is essentially a digest of Ref. [1]

A. Superconductors of Type II

Superconductors are an electronic phase of matter that is characterized by vanishing resistance and the Meissner effect. In the simplest model of superconductivity, electrons condense into pairs, so-called "Cooper paris", when the temperature drops below a critical temperature T_c. Near the transition point, the static condensate is described by the action

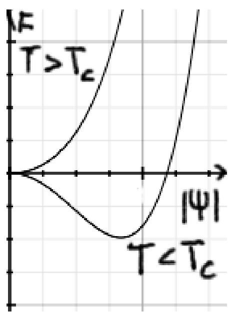
$$S_{GL}[\Psi, \mathbf{A}, \mathbf{H}_{ext}] = \frac{1}{T} \int d^2x \left[\alpha(T)|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{1}{2m^*} \left| \left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A} \right) \Psi \right|^2 + \frac{\mathbf{B}^2}{8\pi} - \frac{\mathbf{B} \cdot \mathbf{H}_{ext}}{4\pi} \right] \tag{1}$$

with parameters

$$\alpha(T) = \alpha(T - T_c), \quad \alpha, \beta \text{ and } m^* \text{ only weakly dependent on } T.$$

This is the celebrated *Ginzburg-Landau action*. Here, $\Psi(x)$ is a complex-valued wave function that describes the condensate. We have also included the dependence on the total magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ and a magnetic source field $\mathbf{H}_{ext}$ which is generated by external currents.
To gain some intuition, let us assume that the condensate is constant in space, $|\Psi(x)| = |\Psi| = \text{const.}$ and let us determine the value that minimizes the action. (We also ignore the magnetic field for now). Above the critical temperature, the action has a single minimum at $|\Psi| = 0$. However, when the temperature is below the critical temperature, the action has a minimum at

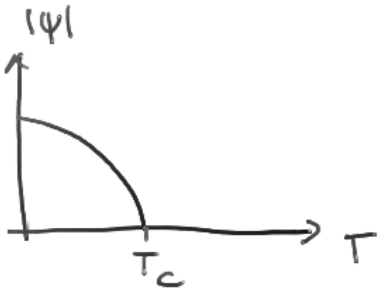
$$\alpha(T)|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 \to \min \implies |\Psi|^2 = -\frac{\alpha(T)}{\beta} = \frac{\alpha T_c}{\beta}\left(1 - \frac{T}{T_c}\right).$$



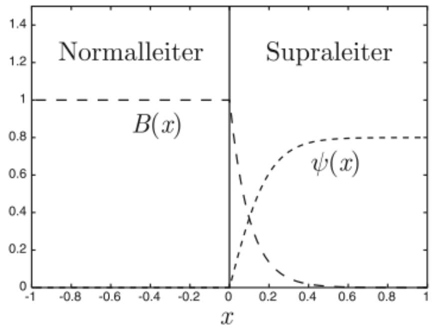
In other words, the temperature dependence of the order parameter is

$$|\Psi| \sim \left(1 - \frac{T}{T_c}\right)^{1/2} \quad \text{for } T < T_c.$$

As the temperature is lowered, the order parameter Ψ changes continuously from zero to a finite value. Such a continous change is also known as a second order phase transition.



Now, consider the interface between a normal conductor and a superconductor. From the Ginzburg-Landau action (1), it is also possible to calculate the magnetic field $\mathbf{B}$. It turns out that the total magnetic field $\mathbf{B}(x)$ will quickly decay as we enter the superconductor and become zero inside. This is the celebrated Meissner effect: superconductors expel magnetic fields. The decay occurs on a length scale $\lambda = \lambda(T)$ which depends on the parameters α, β, m^*. On the other hand, the condensate wave function $\Psi(x)$ will rise from zero outside to non-zero as we enter the superconductor. This occurs on a different length scale $\xi = \xi(T)$.



The relation between these two length scales determines the type of the superconductor. If the quotient $\kappa := \lambda/\xi$ satisfies $\kappa < \frac{1}{\sqrt{2}}$, then it is energetically favorable for the material to become superconducting as a whole; this is a *type I superconductor*. In this case, superconductivity is only possible below a critical magnetic field, $B < B_{c1}$. However, if the quotient satisfies $\kappa > \frac{1}{\sqrt{2}}$, then the magnetic field can penetrate deeper into the superconductor without disturbing the condensate. This means that the material can be superconducting in a higher magnetic field $B_{c1} < B < B_{c2}$ by building as much surface area between the normal conducting and the superconducting phase as possible; this is a *type II superconductor*.

We can understand this with the following very rough estimate: When the condensate wave function is still small in a region of length ξ, then the free energy differs by

$$\approx -A\xi\left(\alpha(T)|\Psi|^2 + \frac{\beta}{2}|\Psi|^4\right) = +A\xi\frac{\alpha(T)^2}{2\beta}$$

from its optimal value. Here, A is the area of the surface between normal and superconducting phase. On the other hand, a magnetic field penetrating in a region of length λ changes the free energy by the amount

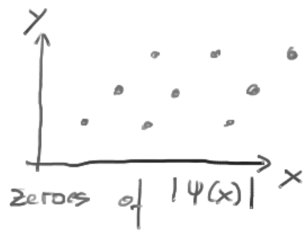
$$\approx -A\lambda\frac{B^2}{8\pi}.$$

When $\xi \ll \lambda$, the second term dominates the first and it is favorable to build as much surface area as possible. A more detailed calculation is necessary to get the precise value for the qutoient κ that distinguishes these two types.

How does a type II superconductor build as much surface area as possible? Abrikosov [2] has shown that the condensate wave function $\Psi(x)$ will take the form of a *vortex lattice*. Its absolute value is periodic with lattice vectors a_1, a_2

$$|\Psi(x + na_1 + ma_2)|^2 = |\Psi(x)|^2,$$

and the complex phase $\chi(x)$ as given by $\Psi(x) = e^{i\chi(x)}|\Psi(x)|$ will change by 2π whenever we encircle a lattice point. It turns out that the triangular lattice is the energetically most favorable configuration.



B. Holographic Superconductors

We now wish to study type II superconductors within the framework of the AdS/CFT correspondence. For simplicity, we restrict our attention to the case of a superconductor confied to two spatial dimensions x, y as opposed to three spatial dimensions, as this case already captures the relevant physics. We mainly give an overview of the results of Ref. [1].

In a previous talk during this seminar, we have already started discussing the AdS/CFT correspondence for superconductors. The idea was to consider a scalar field Ψ and a Maxwell field A_μ in a metric background given by a Reissner-Nordström black brane. The extra dimension was labeled z and had a horizon at $z = z_H$, which indicates the temperature T.

For convenience, we will instead consider the coordinate $u = z/z_H$. Then, the horizon is at $u = 1$ and the conformal boundary is at $u = 0$. The temperature will be explicit in the background metric, which is now given by

$$ds^2 = \frac{L^2\alpha^2}{u^2}(-h(u)dt^2 + dx^2 + dy^2) + \frac{L^2}{u^2 h(u)}du^2$$

$$h(u) = 1 - u^3, \quad \alpha = \frac{4\pi T}{3}.$$

The action for the matter fields is

$$S[\Psi, A_\mu] = \frac{L^2}{2\kappa_4^2 e^2} \int d^4x \sqrt{-g} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - (D_\mu \Psi)^\dagger (D^\mu \Psi) - m^2 \Psi^\dagger \Psi \right).$$

In the probe limit $e \to \infty$, the metric will not be affected by the motion of the matter fields. Variation of the action with respect to the fields gives the equations of motion

$$\frac{1}{\sqrt{-g}} D_\mu(\sqrt{-g} D^\mu \Psi) = m^2 \Psi$$
$$\frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g} F^{\mu\nu}) = j^\nu := i[(D^\nu \Psi)^\dagger \Psi - \Psi^\dagger (D^\nu \Psi)].$$

Boundary conditions. To make sense of these equations, we need to specify boundary conditions. Remember that the conformal boundary $u = 0$ corresponds to the source fields for our partition function. In the presence of a chemical potential μ and a magnetic field B, we have

$$A_t(u = 0) = \mu, \quad F_{xy}(u = 0) = B.$$

For the scalar field Ψ, we focus on the case $m^2 L^2 = -2$, which gives an asymptotic

$$\Psi(u) \sim \Psi_0 u + \Psi_+ u^2 \quad \text{for } u \to 0.$$

In a previous talk, we had discussed that the parameter Ψ_0 corresponds to the external source for the scalar field Ψ. Since we want the field to condense spontaneously, we want the source field to vanish and require that $\Psi_0 = 0$.

At the horizon $u = 1$, we merely require that the field Ψ be finite and that the norm $A_\mu A^\mu$ be finite as well. The reasoning here is that while the horizon is a singularity of the coordinate system, it should be possible to extend the fields continuously to the spacetime beyond the horizon. For the A_t component, this means that

$$A_t A^t = A_t g^{tt} A_t = -\frac{u^2}{L^2 \alpha^2 h(u)} \stackrel{!}{=} \text{finite} \implies A_t(u = 1) = 0.$$

Solution ansatz. We now want to try to solve the equations of motion. Just like in the Ginzburg-Landau theory, we will focus on static configurations, and hence assume that the time derivatives $\partial_t A_\mu$ and $\partial_t \Psi$ vanish.

Moreover, we only consider the system in the vicinity of the critical magnetic field B_{c2}. Hence, we expand the solutions in the small parameter $\varepsilon = (B_{c2} - B)/B_{c2}$ as follows

$$\Psi = \varepsilon^{1/2} \psi_1 + \varepsilon^{3/2} \psi_2 + \dots$$
$$A_\mu = A_\mu^{(0)} + \varepsilon A_\mu^{(1)} + \dots.$$

Remember that we expect a second-order phase transition for the scalar field Ψ, that's why we expect an asymptotics of the form $|\Psi|^2 \sim \varepsilon$.

For the detailed calculation, I have to refer to the paper [1]. However, I still want to give you an impression of how it works. For instance, the equation of motion for the time component A_t becomes

$$\left(\alpha^2 h(u) \frac{\partial^2}{\partial u^2} + \triangle_x \right) A_t = \frac{2L^2 \alpha^2}{u^2} A_t |\Psi|^2$$

in the gauge $A_u = 0$. To lowest order, the solution is simply

$$A_t^{(0)} = \mu(1-u).$$

It interpolates between $A_t(u=1) = 0$ at the horizon and $A_t(u=0) = \mu$ at the conformal boundary.

For the scalar field, we can make a separation ansatz of the form

$$\psi_1(x,u) = \frac{\rho_0(u)}{L}\gamma_L(x)$$

It turns out that at lowest order, there is considerable freedom for the wave function $\gamma_L(x)$. In particular, we can make the ansatz of a vortex lattice. The ansatz by Maeda et. al. [1] is essentially the same as the one by Abrikosov [2].

Effective action and free energy. To find the configuration of lowest energy, we have to calculate the on-shell action to a higher order. The Euler-Lagrange equations tell us that the on-shell action can actually be expressed as an integral over the boundary of our spacetime. Assuming that the superconductor only occupies a finite spatial volume and using the various boundary conditions, it can be shown that the relevant part of the action is the part corresponding to the conformal boundary

$$S - S[\Psi = 0] = \frac{L^2}{2\kappa_4^2 e^2}\frac{\varepsilon^2\alpha}{2}\int d^3x\,\delta^{ij}F_{ui}^{(1)}A_j^{(1)}|_{u=0} + O(\varepsilon^3).$$

Since we are only interested in the specific field configuration Ψ that minimizes the energy, we have subtracted the part of the action that does not depend on the field, $S[\Psi = 0]$.

From the AdS/CFT corresondence, we know that the Maxwell field at the conformal boundary, $A_\mu(u=0)$, is the source field for the current j^μ. We can obtain the expectation value for the current by calculating the functional derivative of the action with respect to the source field:

$$\langle j^k\rangle = \frac{\delta S}{\delta A_k(u=0)} = \frac{L^2}{2\kappa_4^2 e^2}\alpha F_{uk}(u=0).$$

Note that the quantity on the right-hand dise, $F_{uk} = \partial_u A_k$, contains a derivative in the direction of the “extra dimension”. On a technical level, this is how the extra dimension of the AdS/CFT correspondence plays into the calculation.

The action also gives us the thermal free energy. Consider the vortex configuration for the field ψ_1 and write $\sigma(x) = |\gamma_L(x)|^2$. In the limit of long wavelengths, where we only consider variations of the quantity $\sigma(x)$ over large distances, the free energy for this configuration can be calculated to be proportional to

$$F \sim -\varepsilon^2\frac{(\overline{\sigma})^2}{\overline{\sigma^2}}.$$

Here, the notation

$$\overline{f} := \frac{1}{V}\int_V d^2x\, f(x)$$

denotes the spatial average of a quantity f. Minimizing this free energy gives the same result as the Ginzburg-Landau theory: the vortex lattice of the condensate should be triangular.

[1] K. Maeda, M. Natsuume, and T. Okamura, Phys. Rev. D **81**, 026002 (2010). arXiv:0910.4475

[2] A. A. Abrikosov, Sov Phys-JETP (Engl Transl) **5**, 1174 (1957).

[3] G. Czycholl, *Theoretische Festkörperphysik. Von Den Klassischen Modellen Zu Modernen Forschungsthemen*, 3rd ed. (Springer, Berlin, Heidelberg, 2008).